Int J Theor Phys (2008) 47: 1455–1458
DOI 10.1007/s10773-007-9587-8

Lorentz Transformation from Symmetry of Reference Principle

M. Dima · D. Wang

Received: 31 July 2007 / Accepted: 26 September 2007 / Published online: 18 October 2007

Abstract The Lorentz Transformation is traditionally derived requiring the Principle of Relativity and light-speed universality. While the latter can be relaxed, the Principle of Relativity is seen as core to the transformation. The present letter relaxes both statements to the weaker, *Symmetry of Reference* Principle. Thus the resulting Lorentz transformation and its consequences (time dilatation, length contraction) are, in turn, effects of how we manage space and time.

Keywords Lorentz transformation · Symmetry of reference principle

1 Introduction

Starting with the 1905 paper of A. Einstein in *Ann. Phys.* [1] the Lorentz Transformation has been traditionally derived based on the Principle of Relativity and light-speed universality. Various studies (see examples in [2]) have shown that light-speed universality is not needed—the first such publication (1906) being owed to H. Poincaré [3]. Group theory expresses the transitivity property of relativity (C relative to A, if A to B and B to C) in the form of the group closure relation, respectively the product of two group elements being another element of the group. Pure relativity transformations however, cannot form a group on their own, needing rotations to "close" the group. Therefore it is not possible—without more information, to specify the group to which relativity transformations belong to, meaning that the full group is not immediate just from the Principle of Relativity. Traditionally the transformations were specified as belonging to a group that invaries the metric, $\mathbf{\Lambda}^{\dagger}\mathbf{G}\mathbf{\Lambda} = \mathbf{G}$, where $\mathbf{\Lambda}$ are the transformations and $\mathbf{G}$ the metric. This however is an *over*-specification

M. Dima (✉)
Institute for Nuclear Physics and Engineering, Str. Atomistilor No. 407, P.O. Box MG-6, Bucharest—Magurele, Romania
e-mail: modima@theory.nipne.ro

D. Wang
Nuclear Science and Technology Department, Harbin Engineering University, Harbin, Heilongjiang 150001, China

of the physical problem, exceeding the Principle of Relativity. Further more, entering the Lorentz group—i.e. specifying a particular metric (in this case Minkowsky), is equivalent in the end with admitting light speed invariance. In this sense the Principle of Relativity and (indirectly) light speed invariance are core to the Lorentz Transformation [4].

The present letter shows however, that neither statement is necessary and that the Lorentz transformation stems from the simpler (weaker) Principle of Symmetry of reference systems, respectively defining the transformations with the aid of a group is not needed (less so a particular group, say the Lorentz group). This approach shows that the Minkowsky metric is not unique in defining relativity. It will be shown that in fact there are two possible classes of transformations, one invarying the Minkowsky and the other the Euclidean metric. The ad-hoc terminology of Minkowsky and Euclidean relativities will be thus adopted throughout this letter.

2 Space-Time Conditions

Consider two coordinate systems in motion that at some point were at rest relative to each other and were aligned to have the same orientation, offset and (Euclidean) space-metric. The transformation between such coordinate systems is:

$$\begin{pmatrix} dt \\ d\vec{x} \end{pmatrix}' = \underbrace{\begin{pmatrix} \text{scalar}_{1\times 1} & \text{vector}_{1\times 3} \\ \text{vector}_{3\times 1} & \textbf{tensor}_{3\times 3} \end{pmatrix}}_{\mathbf{\Lambda}} \begin{pmatrix} dt \\ d\vec{x} \end{pmatrix} \tag{1}$$

where the dimensions of the objects involved is given by the subscripts. In general the transformation should be an integral, non-linear transformation, however general considerations about space-time limit the range of possible transformations to constant linear transformations:

1. *locality*—implies that the transformation must be point-to-point;
2. *homogeneity*—implies that the transformation must not depend on the relocation of the coordinate system, hence linear;
3. *isotropy*—implies that the mathematical objects in the transformation depend only on $\vec{v}$—the relative velocity between the systems in causa. Dependence on another vector quantity would signal the anisotropy of space. Since $\vec{v}$ is intrinsic to the transformation, the scalars inside it depend only on $|\vec{v}|$, else by rotating the coordinate-system the transformation would look different. The vectors must be parallel to $\vec{v}$, otherwise they introduce preferred directions in space—which in isotropic space do not exist. Likewise, the general form of the tensor is $\lambda\mathbf{C}_\parallel + \mu\mathbf{C}_\perp$, where $\mathbf{C}_\parallel$ selects vector components parallel to $\vec{v}$, and $\mathbf{C}_\perp$ components perpendicular to $\vec{v}$. The tensor cannot contain the $(\times\vec{v})$ anti-symmetric part, for that would change the transformation if the coordinates were chosen as space-time inverted.

Observing these conditions, the transformation can be written as:

$$\mathbf{\Lambda} = \gamma_v \begin{pmatrix} 1 & -\vec{v}/c_v^2 s \\ -\psi_v \vec{v} & \lambda_v \mathbf{C}_\parallel + \mu_v \mathbf{C}_\perp \end{pmatrix} \tag{2}$$

with the scalars fulfilling the roles described above and s a sign factor $s = \pm 1$. For reasons evident later the two shall be termed Minkowsky ($s = +1$) and Euclidean relativity ($s = -1$).

From the above transformation (2) the apparent-velocity law can be derived, for an object moving with $\vec{u}$ in the base-system. This would be seen in the moving-system as:

$$\vec{u} \ominus \vec{v} = \frac{-\psi_v \vec{v} + \lambda_v \vec{u}_\parallel + \mu_v \vec{u}_\perp}{1 - \vec{v}\vec{u}/c_v^2 s} \tag{3}$$

the scalar $c_v = f(v)$ having units of speed.

3 Relativity Conditions

To further define the transformation, a set of relativity conditions are imposed. The following are evident:

1. For $\vec{v} \to 0$ the transformation is unitary:

$$\lim_{v\to 0} \mathbf{\Lambda}_{\vec{v}} = \mathbf{1} \tag{4}$$

   hence $\gamma_v, \lambda_v, \mu_v = 1$.
2. $\vec{v} \ominus \vec{0} = -\vec{0} \ominus \vec{v}$ thus $\psi_v = 1$.
3. $\vec{v} \ominus \vec{v} = \vec{0}$ thus $\lambda_v = \psi_v = 1$.
4. $\mathbf{\Lambda}_{-\vec{v}} = \mathbf{\Lambda}_{\vec{v}}^{-1}$ thus:
   - $\gamma_v = \pm 1/\sqrt{1 - \vec{v}^2/c^2 s}$, the valid sign $(+)$ being determined from $\lim_{v\to 0}$,
   - $\mu_v = \pm 1/\gamma_v$, the valid sign $(+)$ being determined from $\lim_{v\to 0}$.
5. $|\vec{v}_2 \ominus \vec{v}_1| = |\vec{v}_1 \ominus \vec{v}_2|$ implies $c_1 = c_2 = c$ a constant and $s_1 = s_2 =$ the same sign.

4 Discussion

From the apparent-velocity law, the combined speed is:

$$|\vec{u} \ominus \vec{v}|^2 = \frac{(\vec{u} - \vec{v})^2 - s(\vec{u} \times \vec{v})^2}{(1 - \vec{u}\vec{v}/c^2 s)^2} \tag{5}$$

confirming $v_{\lim} \le c$ for $s = +1$ for point (4) above. For both Minkowsky and Euclidean relativity the combined speed $|\vec{u} \ominus \vec{v}|/c$ is shown in Fig. 1—the black bands depicting values above c. Since both cases are symmetric about the first diagonal, only half of each was plotted in order to accommodate the other in the plot for comparison. The two cases are somewhat similar for v or $u > c$, but differ significantly for v and $u < c$.

The transformation is thus now:

$$\mathbf{\Lambda} = \begin{pmatrix} \gamma_v & -\gamma_v \vec{v}/c^2 s \\ -\gamma_v \vec{v} & \gamma_v \mathbf{C}_\parallel + \mathbf{C}_\perp \end{pmatrix} \tag{6}$$

respectively the well known Lorentz transformation for $s = +1$. The meaning of c is related to *causality*: for both Minkowsky and Euclidean relativities the transformed time interval versus proper time is $dt' = \gamma_v(1 - \vec{v}\vec{u}/c^2 s)d\tau$, respectively a *causal* transformation for $v < c$. For $c \to \infty$ the Galilean transformation is recovered.

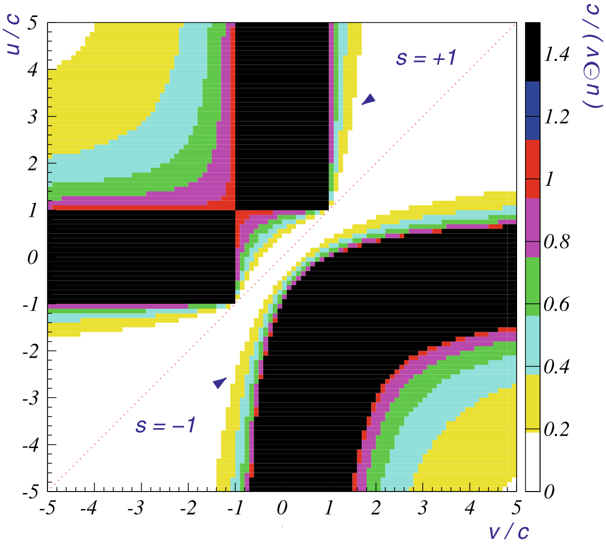

Fig. 1 1D combined speed $|\vec{u} \ominus \vec{v}|/c$ for Minkowsky (*top*) and Euclidean (*bottom*) relativity. The two cases are symmetric about the first diagonal, hence one half has been suppressed for each in order to be plotted together for comparison. The black bands depict values above c

The Principle of Relativity (group "closure"-modulo a rotation) has not been used thus far:

$$\mathbf{\Lambda}_1 \mathbf{\Lambda}_2^{-1} = \mathbf{R}\mathbf{\Lambda}_{12} \tag{7}$$

where $\mathbf{\Lambda}_{1,2}$ are two coordinate transformations, $\mathbf{\Lambda}_{12}$ the system-1 to system-2 transformation and $\mathbf{R}$ an alignment rotation $(\vec{v}_2 \ominus \vec{v}_1) = -\mathbf{R}(\vec{v}_1 \ominus \vec{v}_2)$ that appears due to boost when referencing is done via alignment with a (third party) base-system. In a strong sense the Principle of Relativity is the group "closure" [3] relation (7)—satisfied by both Euclidean and Minkowsky relativities, however in a weaker form it has been used in relations (2), (3) and (5) as the *Symmetry of Reference* Principle. The only isotropic metrics invariant under the transformations are the Euclidean ($s = -1$) and the Minkowsky ($s = +1$) metric.

Acknowledgement One of us (Dima) is thankful for the hospitality during completion of this work to the Nuclear Science and Technology Department of the Harbin Engineering University—China under a Foreign Expert grant.

References

1. Einstein, A.: Ann. Phys. **17**, 891 (1905)
2. Levy-Leblond, J.M.: Am. J. Phys. **44**, 271 (1976)
3. Poincaré, H.: Rend. Circ. Mat. Palermo **21**, 129 (1906)
4. Berzi, V., Gorini, V.: J. Math. Phys. **44**, 271 (1976)